Dynamics of Dark Solitons in Elongated Bose-Einstein Condensates

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We find two types of moving dark soliton textures in elongated condensates: *nonstationary kinks* and *proper dark solitons*. The latter have a flat notch region and we obtain the diagram of their dynamical stability. At finite temperatures the dynamically stable solitons decay due to the thermodynamic instability. We develop a theory of their dissipative dynamics and explain experimental data.

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Recently, several spectacular experiments have demonstrated the creation of vortices [1–3] and dark solitons [4–7] in Bose-Einstein condensates (BEC) of trapped alkali-atom gases. These experiments open an unprecedented possibility to study the dissipative dynamics of such macroscopically excited Bose-condensed states. The vortex has a topological charge (circulation), and the dynamically stable (single-charged) vortices can decay only when reaching the border of the condensate. At finite temperatures in nonrotating traps, the motion of the vortex to the border is induced by thermal excitations and is rather slow. The lifetime of vortices in trapped condensates [8,9] is thus relatively long and extends to a few seconds in the experiments [1–3].

Dark solitons have a density dip and a phase slip in one direction and, as well as vortices, they are particular solutions of the Gross-Pitaevskii (GP) equation. Extensive studies of dark solitons in nonlinear optics [10] have expounded their transverse dynamical instability in 3D, leading to the undulation of the soliton plane and decay into vortex-antivortex pairs and phonon waves. This scenario is similar to that observed at NIST [4] and at Harvard [7]. The decay of solitons into vortex rings was observed at JILA [6]. The transverse instability of dark solitons can be suppressed by a strong radial confinement of their motion in elongated traps [11]. Dynamically stable solitons in such traps are not, however, thermodynamically stable, and their dissipative dynamics is expected to be fundamentally different from that of vortices.

In contrast to vortices, the soliton has no topological charge and can decay without reaching the border of the condensate. Dark solitons behave as objects with a negative mass. The scattering of thermal excitations from the soliton decreases its energy, and the soliton accelerates towards the speed of sound, gradually loses its contrast, and ultimately disappears. This mechanism has been proposed in Ref. [12], and the lifetime of the soliton has been obtained in terms of the reflection coefficient of the excitations. However, the reflection coefficient is calculated wrongly in Ref. [12]. In the 1D case the GP equation is integrable and the reflection vanishes in the Bogolyubov approach. Thus, one expects very long lifetimes of solitons in this limit. On the other hand, in 3D elongated traps the GP equation is no longer integrable and the scattering of thermal excitations from the soliton should be efficient. The absence of topological charge can then lead to a much faster dissipative dynamics of solitons than that of vortices. The Hannover results [5] indeed suggest that moving solitons in a cigar-shaped trap are dynamically stable, but their contrast rapidly decreases to zero due to the thermodynamic instability.

In this Letter we study the dynamics of moving dark solitons in 3D elongated Bose-Einstein condensates and present three important results: (i) We find that using the "phase imprinting method" [5,13] one can generate at least two kinds of soliton textures: *nonstationary kinks* and *proper dark solitons*. The former have a notch region that moves with radially nonuniform velocity and undergoes bending similar to that observed at NIST [4]. These textures are dynamically unstable and decay via the emission of phonons and/or proper dark solitons. The proper solitons are characterized by a flat notch region and can be dynamically stable. Then they propagate without changing their shape; (ii) We derive the diagram of dynamical stability for proper solitons; (iii) We solve the problem of reflection of excitations from the soliton and analyze its decay due to thermodynamic instability. The dissipative dynamics exhibits an interplay between the extent of nonintegrability and the absence of topological charge, and the soliton lifetime ranges from milliseconds for Hannover-type 3D solitons to more than seconds in quasi-1D geometries.

We consider a condensate with repulsive interaction (the scattering length $a > 0$). The condensate wave function can be written as $\Psi(\mathbf{r}, t)\exp(-i\mu t)$, where μ is the chemical potential. In an infinitely long cylindrical harmonic trap this function satisfies the GP equation

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}=\left\{-\frac{\hbar^2}{2m}\Delta+\frac{m}{2}\omega_\rho^2\rho^2+g|\Psi(\mathbf{r},t)|^2-\mu\right\}\Psi(\mathbf{r},t) \quad (1)$$

Here ω_ρ is the frequency of the radial (ρ) confinement, $g=4\pi\hbar^2a/m$, and m is the atom mass. The wave function of the ground-state condensate minimizes the corresponding energy functional and is the solution of Eq. (1) with zero left-hand side. Macroscopically excited BEC states (solitons, vortices) do not correspond to the minimum of the energy functional and are thermodynamically unstable.

A stationary, or solitary-wave macroscopically excited BEC state can also be dynamically unstable with regard to elementary excitations around it. The unstable excitation modes are characterized by complex eigenfrequencies and grow exponentially in time, which indicates that the BEC state will evolve far from the initial shape.

Strictly speaking, dark solitons are solutions of the 1D GP equation in free space. They are characterized by a local density minimum (notch) moving with a constant velocity v, and by a phase gradient of the wave function at the position of the minimum. In an otherwise uniform condensate of density n_0, the dark soliton state is described by the wave function (see [12] and references therein)

$$\Psi(z,t)=\sqrt{n_0}(\cos\theta-i\sin\theta\tanh[\sin\theta(z-vt)/l_0]), \quad (2)$$

where $\cos\theta=v/c_s$, and $c_s=\sqrt{n_0g/m}$ is the speed of sound. The quantities θ and $-\theta$ are the phases of the soliton state at $(z-vt)\to-\infty$ and $(z-vt)\to\infty$, respectively. The width L of the notch (soliton plane) is of the order of the correlation length $l_0=\hbar/mc_s$. Approximate soliton solutions can be also found in 1D harmonic traps [14,15]. Solitons which have nonzero velocity in the center of the trap oscillate along the trap axis [15].

In 3D harmonic traps the solutions of the GP equation, describing standing dark solitons ($v=0$ and $\partial\Psi/\partial t=0$), have been found in [11,16]. For infinitely long cylindrical condensates these solutions follow from Eq. (1) and can decay due to the transverse dynamical instability. The stability criterion requires a strong radial confinement providing a non-Thomas-Fermi (TF) regime with the radial size of the condensate $r\lesssim L\sim l_0$.

The existence of moving solitonlike textures in 3D elongated condensates is confirmed by the experiments and simulations [5], but no analytical solution has been found so far. In the TF regime ($\mu\gg\hbar\omega_\rho$), one can use Eq. (2) with ρ-dependent n_0 and l_0. Then, the absence of the radial flux of particles at an infinite axial separation from the notch requires the phase θ to be independent of the radial coordinate. This means that the notch velocity v depends on ρ and is proportional to the local velocity of sound $c_s(\rho)$. Hence, the central regions of the notch move faster than the borders, and an initially flat notch region bends in the course of motion. Our simulations for TF condensates show that imprinting of a π-phase slip along z axis with ρ-independent optical potential creates such *nonstationary kinks.* They are dynamically unstable and decay on a time scale of the order of ω_ρ^{-1}. The notch surface bends, whereas the notch velocity increases and the depth decreases. For $\mu/\hbar\omega_\rho>10$ the nonstationary kink decays ultimately into phonon waves. For $\mu/\hbar\omega_\rho\lesssim 5$ it transforms into a *proper dark soliton* characterized by a flat notch region and ρ-independent velocity. In non-TF condensates ($\mu\sim\hbar\omega_\rho$) we generated the proper solitons directly by simulating the π-phase slip imprinting.

We first develop an analytical approach for describing moving proper solitons in the limiting case where the axial size L of the soliton notch greatly exceeds the radial size r of the condensate, and the solitons are expected to be dynamically stable. In an infinitely long cylindrical condensate, the velocity v and the shape of a dynamically stable soliton remain unchanged in the absence of dissipation. The wave function Ψ depends on ρ and $x=z-vt$ and we write it in the form $\Psi(\rho,x)=\psi(\rho,x)f(x)$, where the functions $\psi(\rho,x)$ and $f(x)$ satisfy the equations

$$i\hbar\partial\psi/\partial t=\{-(\hbar^2/2m)[\Delta+2(\nabla_xf/f)\nabla_x]+m\omega_\rho^2\rho^2/2+g|f|^2|\psi|^2-\tilde{\mu}(|f|)\}\psi, \quad (3)$$

$$i\hbar\partial f/\partial t=-(\hbar^2/2m)\Delta_xf+[\tilde{\mu}(f)-\mu]f. \quad (4)$$

The quantity $\tilde{\mu}$ is a functional of f and has to be found self-consistently from Eqs. (3) and (4). We will select the function $\psi(\rho,x)$ such that at infinite x it becomes the wave function of the ground-state condensate, $\psi_0(\rho)$. Hence, for $|x|\to\infty$ we have $|f|\to 1$ and $\tilde{\mu}\to\mu$.

Under the condition $r\ll L$ the radial distribution of particles is close to that for the ground-state condensate. The quantity $\tilde{\mu}(f)$ is close to μ and can be expressed as $\tilde{\mu}(f)=\mu+(|f|^2-1)g\partial\mu/\partial g+\delta\tilde{\mu}$, where $\delta\tilde{\mu}$ is a correction of higher order in r/L. The quantity $g\partial\mu/\partial g=m\bar{c}_s^2$, where $\bar{c}_s$ is nothing else than the velocity of axially propagating sound waves in the ground-state condensate. In the quasi-1D regime, where the interparticle interaction at maximum condensate density $n_{0m}g\ll\hbar\omega_\rho$, we have an almost Gaussian density profile. The radial size $r\sim l_\rho=(\hbar/m\omega_\rho)^{1/2}$, and the small parameter of the expansion for $\tilde{\mu}$ is $(r/L)^2\sim n_{0m}g/\hbar\omega_\rho$. To first order in $n_{0m}g/\hbar\omega_\rho$ we obtain $g\partial\mu/\partial g=n_{0m}g/2$. This gives the velocity $\bar{c}_s$ by a factor of $\sqrt{2}$ smaller than the speed of sound at maximum density: $\bar{c}_s=\sqrt{n_{0m}g/2m}$. For radially TF condensates the chemical potential $\mu\propto\sqrt{g}$ and we arrive at the same expression for $\bar{c}_s$. This result for TF elongated condensates has been obtained in [17] and found in the MIT experiment [18]. The condition $r\ll L$ requires fast TF solitons for which the density dip is small and the function $|f(x)|$ is close to 1.

Omitting the higher order correction $\delta\tilde{\mu}$, Eq. (4) for the function $f(x)$ becomes an ordinary 1D GP equation

$$i\hbar\partial f/\partial t = -(\hbar^2/2m)\Delta_x f + m\bar{c}_s^2[(|f|^2 - 1)f. \quad (5)$$

The dark soliton solution $f(x)$ is given by the right-hand side (rhs) of Eq. (2), where c_s is replaced by $\bar{c}_s$, l_0 by $\bar{l}_0 = l_0/\sqrt{2}$, and n_0 by unity. The solitonlike wave function of the condensate can be written as $\Psi(\rho, x) = \psi_0(\rho)f(x)$. For $v \to \bar{c}_s$ the condition $r \ll L$ is always satisfied, and one clearly sees that in any case the maximum soliton velocity (at which the soliton disappears) is equal to $\bar{c}_s$.

The dynamical stability of proper solitons is determined by the parameter r/L, and the instability border is reached for $r \sim L$. In this case the developed analytical approach can be no longer used and we have found the proper soliton solutions numerically from Eq. (1). To investigate the dynamical stability of moving solitons we have numerically solved a time-dependent equation for elementary excitations around the obtained soliton wave function [19]. For a given soliton velocity v, the parameter r/L increases with the ratio $n_{0m}g/\hbar\omega_\rho$. Above a critical value $n_{0m}g/\hbar\omega_\rho = \xi_c$ (r/L close to 1) the transverse instability was manifesting itself in our calculations as a dramatic rise of excitation modes. In Fig. 1 we present the critical ratio ξ_c as a function of the soliton velocity. For $n_{0m}g/\hbar\omega_\rho < \xi_c$ the solitons are dynamically stable. Note that $\xi_c = 2.5$ for a standing soliton, which agrees with the earlier calculation [11]. For the solitons with high velocities the relation $r \sim L$ is reached at larger $n_{0m}g/\hbar\omega_\rho$, and thus the stability condition is more relaxed [20].

We now discuss the thermodynamic instability of dynamically stable solitons in the presence of a thermal cloud and start with the quasi-1D regime ($n_{0m}g \ll \hbar\omega_\rho$). In this regime we have $r \ll L$ and the dark solitons are described by Eq. (5) which is completely integrable. Hence, the solitons are transparent for excitations. This follows directly from the solutions for time-dependent excitations of the solitonlike condensate [22]. Thus there is no energy and momentum exchange between the soliton and the thermal cloud, and the thermodynamic instability does not manifest itself.

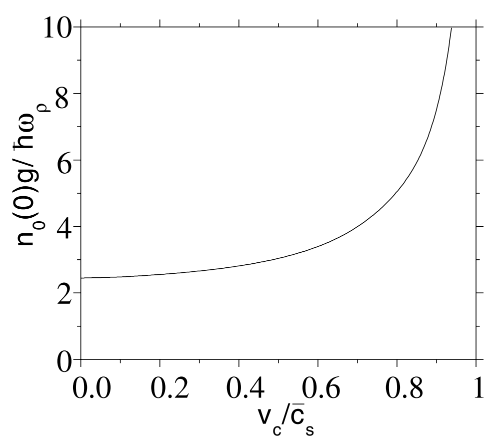

FIG. 1. The critical ratio $n_{0m}g/\hbar\omega_\rho = \xi_c$ versus the soliton velocity v (in units of $\bar{c}_s$). Solitons are stable below this curve.

The dissipative dynamics of the solitons originates from the interaction between the radial and axial degrees of freedom, which results in the second order correction $\delta\tilde{\mu}$ to the quantity $\tilde{\mu}$ in Eq. (4). In the quasi-1D regime, to first order in $n_{0m}g/\hbar\omega_\rho$ the function $\psi(\rho, x) = \psi_0(\rho) + \delta\psi(\rho, x)$, where a small term $\delta\psi$ is real. The correction $\delta\tilde{\mu}$ is then equal to $(-\gamma n_{0m}g/2)(|f|^2 - 1)]^2$, where $\gamma = 3n_{0m}g\ln(4/3)/\hbar\omega_\rho \ll 1$. Thus, we obtain $\tilde{\mu}(f) = \mu + m\bar{c}_s^2[(|f|^2 - 1) - \gamma(|f|^2 - 1)^2]$, and Eq. (4) becomes

$$i\hbar\frac{\partial f}{\partial t} = -\frac{\hbar^2}{2m}\frac{d^2 f}{dx^2} + m\bar{c}_s^2[(|f|^2 - 1) - \gamma(|f|^2 - 1)^2]f. \quad (6)$$

The interaction between the axial and radial degrees of freedom is described by the small term $\gamma(|f|^2 - 1)^2$ in the rhs of Eq. (6). This term only slightly modifies $\Psi(\rho, x)$, but it lifts the integrability of the equation and leads to the reflection of excitations from the soliton.

For finding the probability of reflection of an incident excitation with momentum k and energy ε [reflection coefficient $R(k)$], we have solved the Bogolyubov–de Gennes equations following from Eq. (6). For the phonon branch of the spectrum, where the axial momentum of an excitation $k \ll \bar{l}_0^{-1}$, the excitation wave functions u, v were found in the form of expansion in powers of $k\bar{l}_0$ and γ around the fundamental modes of the Bogolyubov–de Gennes equations with $\gamma = 0$. The u, v functions of these equations were obtained straightforwardly for an arbitrary k and used for calculating the reflection coefficient $R(k)$ from the Fermi golden rule. For $k\bar{l}_0 \ll 1$ the obtained $R(k)$ matches the one following from the method of fundamental modes. Thus at any ε and k we obtain

$$R(k) = \left[\frac{8\pi\gamma(\varepsilon - \hbar k v)\bar{l}_0^2}{9\hbar\sinh\{(\pi(|k| + |k'|)\bar{l}_0/2\sqrt{1 - v^2/\bar{c}_s^2}\}}\right]^2 \frac{|kk'|}{\nu(k)\nu(k')}, \quad (7)$$

where $\nu(k) = \partial(\varepsilon - \hbar k v)\partial\hbar k$ is the group velocity. The energy ε' and momentum k' of the reflected wave are related to ε and k by the energy conservation law in the reference frame moving together with the soliton, $\varepsilon - \hbar k v = \varepsilon' - \hbar k' v$. For small k the reflection coefficient increases as $k^2 \propto \varepsilon^2$. The coefficient reaches its maximum at $k \sim \bar{l}_0^{-1}\sqrt{1 - v^2/\bar{c}_s^2}$ and decays exponentially for large k.

The reflection of excitations from the soliton provides a momentum transfer from the thermal cloud to the soliton. Hence, there is a friction force acting on the soliton. The momentum transfer per unit time is given by

$$\dot{p} = \int_{-\infty}^{\infty} \hbar(k - k')R(k)\nu(k)N(\varepsilon - \hbar k v)dk/2\pi, \quad (8)$$

where $N(\varepsilon - \hbar k v)$ are excitations occupation numbers.

The energy of the soliton can be written in the form $H = (M\bar{c}_s^2/3)(1 - v^2/\bar{c}_s^2)^{3/2}$, with $M = 2\pi n_{0m}\bar{l}_0 r^2 m$ being the effective mass of the soliton ($r = \sqrt{2}l_\rho$ in the quasi-1D regime). The soliton energy decreases with increasing v. For example, for $v \ll \bar{c}_s$ we have $H = M\bar{c}_s^2/3 - Mv^2/2$. The quantity $N_* = M/m \gg 1$ is the number of particles that one has to remove from the condensate in order to create the soliton density dip. Thus, the dark soliton can be treated as a heavy classical particlelike object with a negative mass, and the friction force accelerates the soliton towards the velocity of sound (see [12]).

The Hamiltonian equation $\partial H/\partial p = v$ gives $\dot{p} = -M\dot{v}(1 - v^2/\bar{c}_s^2)^{1/2}$. Then, from Eqs. (7) and (8) we obtain the time dependence of the soliton velocity and that of the soliton contrast $C = (1 - v^2/\bar{c}_s^2)$. This immediately gives the time t at which the contrast decreases from the initial value C_0 to $C(t)$. In particular, for $T \gg n_{0m}g$ this time can be found from the relations

$$F(C) - F(C_0) = t/\tau, \qquad \tau = \hbar N_*/TR_0. \tag{9}$$

Here $F(C)$ is a universal function of the contrast, and $R_0 = 0.084\gamma^2$ is the maximum value of the reflection coefficient in the limit of $v \to 0$. The function $F(C)$ was calculated numerically and can be approximated as $F(C) \approx 0.47\ln\{(1 - C)/C\}$ [23]. The quantity τ can be regarded as a characteristic lifetime of the soliton. For example, if the contrast is initially equal to 30%, it decreases to 10% at a time $t \approx 0.25\tau$.

The dissipative dynamics of quasi-1D solitons is governed by the small extent of nonintegrability of Eq. (6) ($\gamma \ll 1$) and the time τ can be very long. The physical picture changes to the opposite one if the axial size L of the notch becomes comparable with the radial size r of the condensate. Then the nonintegrability of the GP equation is essential and the absence of topological charge provides a fast dissipative dynamics.

The condition $L \sim r$ was fulfilled for solitons in the Hannover experiment [5]. For the number of atoms $N \approx 1.5 \times 10^5$, and the trap frequencies $\omega_z = 2\pi \times 14$ Hz and $\omega_\rho = 2\pi \times 425$ Hz, we calculate the critical temperature $T_c \approx 350$ nK, the maximum density $n_{0m} \approx 4 \times 10^{14}$ cm^{-3}, and the chemical potential $\mu \approx 140$ nK. This indicates that the solitons were in the TF regime, with $\mu/\hbar\omega_\rho \approx 7$. The thermal fraction was about 10%, which corresponds to $T \approx 0.5T_c$ and $\mu/T \approx 0.8$. The measured soliton contrast was decreasing from $\approx 30\%$ to $\approx 10\%$ at a time of 15 (± 5) ms. The results in Fig. 1 indicate that the dark solitons of Ref. [5] were dynamically stable. Using a direct numerical approach, we calculated the reflection coefficient of excitations from the soliton. The dependence of R on k and v is similar to that in the limit of $r \ll L$. The maximum reflection coefficient for $v \to 0$ is $R_0 \approx 0.7$. Then from Eq. (9) we find $\tau \approx 80$ ms and conclude that the soliton contrast decreases from 30% to 10% at a time of 20 ($\pm$ 5) ms, in agreement with [5].

In conclusion, we have investigated the dynamical stability and dissipative dynamics of solitons in elongated BEC's, and explained the experimental data of Ref. [5]. For recently achieved quasi-1D BEC's, [24] the parameter $\gamma \sim 0.1$ and our theory predicts the soliton lifetime larger than seconds. This opens prospects for studying dissipative phenomena originating from the quantum character of the boson field omitted in the common GP approach. Temperature dependence of the soliton lifetime offers interesting possibilities of BEC thermometry.

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